EM Algorithm for Latent Variable Models

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General Latent Variable Model

- Two sets of random variables: z and x.
- z consists of unobserved **hidden variables**.
- x consists of **observed variables**.
- Joint probability model parameterized by $\theta \in \Theta$:

$$p(x, z \mid \theta)$$

Complete and Incomplete Data

- An observation of x is called an **incomplete data set**.
- An observation (x, z) is called a **complete data set**.

- Suppose we have an incomplete data set $\mathcal{D} = (x_1, \dots, x_n)$.
- To simplify notation, take x to represent the entire dataset

$$x = (x_1, \dots, x_n)\,,$$

and z to represent the corresponding unobserved variables

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Our Objectives

- Given incomplete dataset $\mathcal{D} = x = (x_1, \ldots, x_n)$, find MLE

$$\hat{\theta} = \underset{\theta}{\arg\max}\, p(\mathcal{D} \mid \theta).$$

- Given x_i, find conditional distribution over z_i:

$$p\left(z_i \mid x_i, \theta\right).$$

  - For Gaussian mixture model, this is easy once we have θ:

$$\begin{aligned} p(z_i \mid x_i, \theta) &\propto p(z_i, x_i \mid \theta_i) \\ &= \pi(z_i)\mathcal{N}\left(x_i \mid \mu_{z_i}, \Sigma_{z_i}\right). \end{aligned}$$

Lower Bound for Likelihood

- Let $q(z)$ be any PMF on $\mathcal{Z}$, the support of Z:

$$
\begin{aligned}
\log p(x \mid \theta) &= \log\left[\sum_z p(x, z \mid \theta)\right] \\
&= \log\left[\sum_z q(z)\left(\frac{p(x, z \mid \theta)}{q(z)}\right)\right] \quad \text{(log of an expectation)} \\
&\geqslant \underbrace{\sum_z q(z) \log\left(\frac{p(x, z \mid \theta)}{q(z)}\right)}_{\mathcal{L}(q, \theta)} \quad \text{(expectation of log)}
\end{aligned}
$$

- We'll maximize the lower $\mathcal{L}(q, \theta)$ w.r.t. θ and q.

Lower Bound and Expected Complete Log-Likelihood

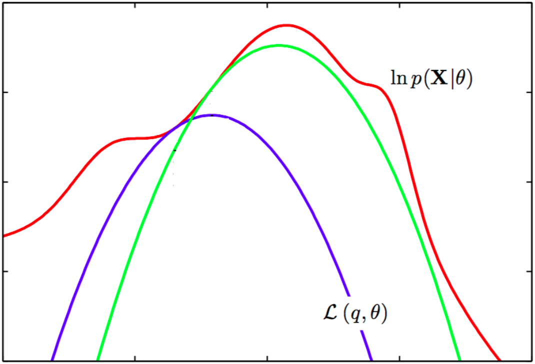
- Consider maximizing the lower bound $\mathcal{L}(q,\theta)$:

$$\begin{aligned}\mathcal{L}(q,\theta) &= \sum_z q(z)\log\left(\frac{p(x,z\mid\theta)}{q(z)}\right)\\ &= \underbrace{\sum_z q(z)\log p(x,z\mid\theta)}_{\mathbb{E}[\text{complete data log-likelihood}]} - \underbrace{\sum_z q(z)\log q(z)}_{\text{no } \theta \text{ here}}\end{aligned}$$

- Maximizing $\mathcal{L}(q,\theta)$ equivalent to maximizing $\mathbb{E}$[complete data log-likelihood].

A Family of Lower Bounds

- Each q gives a different lower bound: $\log p(x \mid \theta) \geqslant \mathcal{L}(q, \theta)$
- Two lower bounds, as functions of θ:

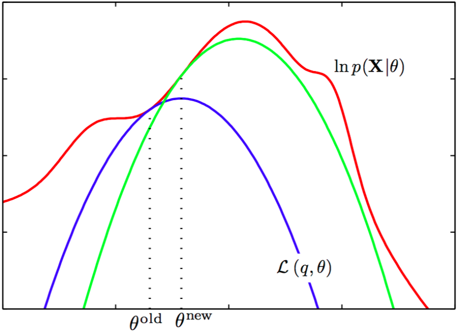


From Bishop's *Pattern recognition and machine learning*, Figure 9.14.

EM: Coordinate Ascent on Lower Bound

- Choose sequence of q's and θ's by "**coordinate ascent**".
- EM Algorithm (high level):
  1. Choose initial θ^{old}.
  2. Let $q^* = \arg\max_q \mathcal{L}(q, \theta^{\text{old}})$
  3. Let $\theta^{\text{new}} = \arg\max_\theta \mathcal{L}(q^*, \theta^{\text{old}})$.
  4. Go to step 2, until converged.
- Will show: $p(x \mid \theta^{\textbf{new}}) \geqslant p(x \mid \theta^{\textbf{old}})$
- **Get sequence of θ's with monotonically increasing likelihood.**

EM: Coordinate Ascent on Lower Bound



1. Start at θ^{old}.
2. Find q giving best lower bound at $\theta^{\text{old}} \implies \mathcal{L}(q,\theta)$.
3. $\theta^{\text{new}} = \arg\max_{\theta} \mathcal{L}(q,\theta)$.

From Bishop's *Pattern recognition and machine learning*, Figure 9.14.

The Lower Bound

- Let's investigate the lower bound:

$$
\begin{aligned}
\mathcal{L}(q,\theta) &= \sum_z q(z)\log\left(\frac{p(x,z\mid\theta)}{q(z)}\right) \\
&= \sum_z q(z)\log\left(\frac{p(z\mid x,\theta)p(x\mid\theta)}{q(z)}\right) \\
&= \sum_z q(z)\log\left(\frac{p(z\mid x,\theta)}{q(z)}\right) + \sum_z q(z)\log p(x\mid\theta) \\
&= -\mathrm{KL}[q(z), p(z\mid x,\theta)] + \log p(x\mid\theta)
\end{aligned}
$$

- Amazing! We get back an equality for the marginal likelihood:

$$
\log p(x\mid\theta) = \mathcal{L}(q,\theta) + \mathrm{KL}[q(z), p(z\mid x,\theta)]
$$

The Best Lower Bound

- Find q maximizing

$$\mathcal{L}(q, \theta^{\text{old}}) = -\mathrm{KL}[q(z), p(z \mid x, \theta^{\text{old}})] + \underbrace{\log p(x \mid \theta^{\text{old}})}_{\text{no } q \text{ here}}$$

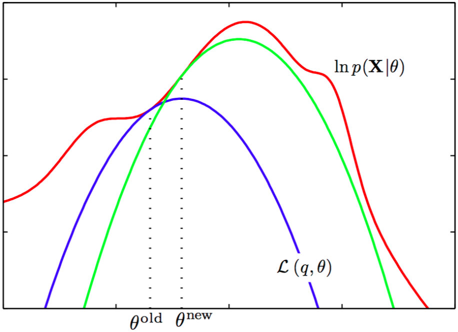
- Recall $\mathrm{KL}(p \| q) \geqslant 0$, and $\mathrm{KL}(p \| p) = 0$.
- Best q is $q^*(z) = p(z \mid x, \theta^{\text{old}})$ and

$$\mathcal{L}(q^*, \theta^{\text{old}}) = -\underbrace{\mathrm{KL}[p(z \mid x, \theta^{\text{old}}), p(z \mid x, \theta^{\text{old}})]}_{=0} + \log p(x \mid \theta^{\text{old}})$$

- Summary:

$$\begin{aligned} \log p(x \mid \theta^{\text{old}}) &= \mathcal{L}(q^*, \theta^{\text{old}}) \quad (\text{tangent at } \theta^{\text{old}}). \\ \log p(x \mid \theta) &\geqslant \mathcal{L}(q^*, \theta) \quad \forall \theta \end{aligned}$$

Tight lower bound for any chosen θ



Fix any θ' and take $q'(z) = p(z \mid x, \theta')$. Then

1. $\log p(x \mid \theta) \geqslant \mathcal{L}(q', \theta)$ $\forall \theta$. [Global lower bound].
2. $\log p(x \mid \theta') = \mathcal{L}(q', \theta')$. [Lower bound is **tight** at θ'.]

From Bishop's *Pattern recognition and machine learning*, Figure 9.14.

General EM Algorithm

1. Choose initial θ^{old}.
2. **Expectation Step**
   - Let $q^*(z) = p(z \mid x, \theta^{\text{old}})$. [$q^*$ gives best lower bound at θ^{old}]
   - Let

$$J(\theta) := \mathcal{L}(q^*, \theta) = \underbrace{\sum_z q^*(z) \log\left(\frac{p(x, z \mid \theta)}{q^*(z)}\right)}_{\textbf{expectation} \text{ w.r.t. } z \sim q^*(z)}$$

3. **Maximization Step**

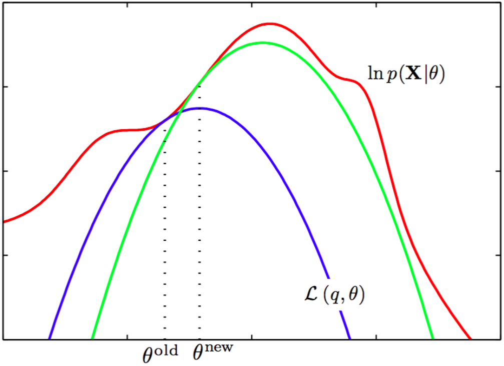
$$\theta^{\text{new}} = \arg\max_\theta J(\theta).$$

   [Equivalent to maximizing expected complete log-likelihood.]

4. Go to step 2, until converged.

EM Monotonically Increases Likelihood

EM Gives Monotonically Increasing Likelihood: By Picture



From Bishop's *Pattern recognition and machine learning*, Figure 9.14.

EM Gives Monotonically Increasing Likelihood: By Math

1. Start at θ^{old}.
2. Choose $q^*(z) = \arg\max_q \mathcal{L}(q, \theta^{\text{old}})$. We've shown

$$\log p(x \mid \theta^{\text{old}}) = \mathcal{L}(q^*, \theta^{\text{old}})$$

3. Choose $\theta^{\text{new}} = \arg\max_\theta \mathcal{L}(q^*, \theta^{\text{old}})$. So

$$\mathcal{L}(q^*, \theta^{\text{new}}) \geqslant \mathcal{L}(q^*, \theta^{\text{old}}).$$

Putting it together, we get

$$\begin{aligned}
\log p(x \mid \theta^{\text{new}}) &\geqslant \mathcal{L}(q^*, \theta^{\text{new}}) && \mathcal{L} \text{ is a lower bound} \\
&\geqslant \mathcal{L}(q^*, \theta^{\text{old}}) && \text{By definition of } \theta^{\text{new}} \\
&= \log p(x \mid \theta^{\text{old}}) && \text{Bound is tight at } \theta^{\text{old}}.
\end{aligned}$$

Suppose We Maximize the Lower Bound...

- Suppose we have found a **global maximum** of $\mathcal{L}(q,\theta)$:

$$L(q^*,\theta^*) \geqslant L(q,\theta)\ \forall q,\theta,$$

  where of course

$$q^*(z) = p(z \mid x,\theta^*).$$

- Claim: θ^* is a global maximum of $\log p(x \mid \theta^*)$.
- Proof: For any θ', we showed that for $q'(z) = p(z \mid x,\theta')$ we have

$$\begin{aligned}
\log p(x \mid \theta') &= \mathcal{L}(q',\theta') + \mathrm{KL}[q', p(z \mid x,\theta')] \\
&= \mathcal{L}(q',\theta') \\
&\leqslant \mathcal{L}(q^*,\theta^*) \\
&= \log p(x \mid \theta^*)
\end{aligned}$$

Convergence of EM

- Let θ_n be value of EM algorithm after n steps.
- Define "transition function" $M(\cdot)$ such that $\theta_{n+1} = M(\theta_n)$.
- Suppose log-likelihood function $\ell(\theta) = \log p(x \mid \theta)$ is differentiable.
- Let S be the set of stationary points of $\ell(\theta)$. (i.e. $\nabla_\theta \ell(\theta) = 0$)

Theorem

Under mild regularity conditions[a], *for any starting point* θ_0,

- $\lim_{n\to\infty} \theta_n = \theta^*$ *for some stationary point* $\theta^* \in S$ *and*
- θ^* *is a fixed point of the EM algorithm, i.e.* $M(\theta^*) = \theta^*$. *Moreover,*
- $\ell(\theta_n)$ *strictly increases to* $\ell(\theta^*)$ *as* $n \to \infty$, *unless* $\theta_n \equiv \theta^*$.

[a] For details, see "Parameter Convergence for EM and MM Algorithms" by Florin Vaida in *Statistica Sinica* (2005). http://www3.stat.sinica.edu.tw/statistica/oldpdf/a15n316.pdf

Variations on EM

EM Gives Us Two New Problems

- The "E" Step: Computing

$$J(\theta) := \mathcal{L}(q^*, \theta) = \sum_z q^*(z) \log\left(\frac{p(x, z \mid \theta)}{q^*(z)}\right)$$

- The "M" Step: Computing

$$\theta^{\text{new}} = \underset{\theta}{\arg\max}\, J(\theta).$$

- Either of these can be too hard to do in practice.

Generalized EM (GEM)

- Addresses the problem of a difficult "M" step.
- Rather than finding
$$\theta^{\text{new}} = \arg\max_{\theta} J(\theta),$$
find **any** θ^{new} for which
$$J(\theta^{\text{new}}) > J(\theta^{\text{old}}).$$
- Can use a standard nonlinear optimization strategy
  - e.g. take a gradient step on J.
- We still get monotonically increasing likelihood.

EM and More General Variational Methods

- Suppose "E" step is difficult:
  - Hard to take expectation w.r.t. $q^*(z) = p(z \mid x, \theta^{\text{old}})$.
- Solution: Restrict to distributions $\mathcal{Q}$ that are easy to work with.
- Lower bound now looser:

$$q^* = \underset{q \in \mathcal{Q}}{\arg\min} \, \mathrm{KL}[q(z), p(z \mid x, \theta^{\text{old}})]$$

EM in Bayesian Setting

- Suppose we have a prior $p(\theta)$.
- Want to find MAP estimate: $\hat{\theta}_{\text{MAP}} = \arg\max_\theta p(\theta \mid x)$:

$$
\begin{aligned}
p(\theta \mid x) &= p(x \mid \theta)p(\theta)/p(x) \\
\log p(\theta \mid x) &= \log p(x \mid \theta) + \log p(\theta) - \log p(x)
\end{aligned}
$$

.

- Still can use our lower bound on $\log p(x, \theta)$.

$$
J(\theta) := \mathcal{L}(q^*, \theta) = \sum_z q^*(z) \log\left(\frac{p(x, z \mid \theta)}{q^*(z)}\right)
$$

- Maximization step becomes

$$
\theta^{\text{new}} = \arg\max_\theta \left[J(\theta) + \log p(\theta)\right]
$$

- Homework: Convince yourself our lower bound is still tight at θ.

Homework: Gaussian Mixture Model (Hints)

Homework: Derive EM for GMM from General EM Algorithm

- Subsequent slides may help set things up.
- Key skills:
  - MLE for multivariate Gaussian distributions.
  - Lagrange multipliers

Gaussian Mixture Model (k Components)

- GMM Parameters

$$\begin{aligned}\text{Cluster probabilities :} \quad & \pi = (\pi_1, \ldots, \pi_k) \\ \text{Cluster means :} \quad & \mu = (\mu_1, \ldots, \mu_k) \\ \text{Cluster covariance matrices:} \quad & \Sigma = (\Sigma_1, \ldots \Sigma_k)\end{aligned}$$

- Let $\theta = (\pi, \mu, \Sigma)$.

- Marginal log-likelihood

$$\log p(x \mid \theta) = \log \left\{ \sum_{z=1}^{k} \pi_z \mathcal{N}(x \mid \mu_z, \Sigma_z) \right\}$$

$q^*(z)$ are "Soft Assignments"

- Suppose we observe n points: $X = (x_1, \ldots, x_n) \in \mathbf{R}^{n \times d}$.
- Let $z_1, \ldots, z_n \in \{1, \ldots, k\}$ be corresponding hidden variables.
- Optimal distribution q^* is:

$$q^*(z) = p(z \mid x, \theta).$$

- Convenient to define the conditional distribution for z_i given x_i as

$$\begin{aligned} \gamma_i^j &:= p\left(z = j \mid x_i\right) \\ &= \frac{\pi_j \mathcal{N}\left(x_i \mid \mu_j, \Sigma_j\right)}{\sum_{c=1}^k \pi_c \mathcal{N}\left(x_i \mid \mu_c, \Sigma_c\right)} \end{aligned}$$

Expectation Step

- The complete log-likelihood is

$$
\begin{aligned}
\log p(x, z \mid \theta) &= \sum_{i=1}^{n} \log \left[\pi_z \mathcal{N}\left(x_i \mid \mu_z, \Sigma_z\right)\right] \\
&= \sum_{i=1}^{n} \left(\log \pi_z + \underbrace{\log \mathcal{N}\left(x_i \mid \mu_z, \Sigma_z\right)}_{\text{simplifies nicely}} \right)
\end{aligned}
$$

- Take the expected complete log-likelihood w.r.t. q^*:

$$
\begin{aligned}
J(\theta) &= \sum_{z} q^*(z) \log p(x, z \mid \theta) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{k} \gamma_i^j \left[\log \pi_j + \log \mathcal{N}\left(x_i \mid \mu_j, \Sigma_j\right)\right]
\end{aligned}
$$

Maximization Step

- Find θ^* maximizing $J(\theta)$:

$$\begin{aligned}
\mu_c^{\text{new}} &= \frac{1}{n_c}\sum_{i=1}^{n}\gamma_i^c x_i \\
\Sigma_c^{\text{new}} &= \frac{1}{n_c}\sum_{i=1}^{n}\gamma_i^c \left(x_i-\mu_{\text{MLE}}\right)\left(x_i-\mu_{\text{MLE}}\right)^T \\
\pi_c^{\text{new}} &= \frac{n_c}{n},
\end{aligned}$$

for each $c = 1, \ldots, k$.