

## Week 2 Pre-Lecture: Concept Check Exercises

### Optimization Prerequisites for Lasso

1. Given  $a \in \mathbb{R}$  we define  $a^+, a^-$  as follows:

$$a^+ = \begin{cases} a & \text{if } a \geq 0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad a^- = \begin{cases} -a & \text{if } a < 0, \\ 0 & \text{otherwise.} \end{cases}$$

We call  $a^+$  the *positive part* of  $a$  and  $a^-$  the *negative part* of  $a$ . Note that  $a^+, a^- \geq 0$ .

- (a) Give an expression for  $a$  in terms of  $a^+, a^-$ .
- (b) Give an expression for  $|a|$  in terms of  $a^+, a^-$ .  
For  $x \in \mathbb{R}^d$  define  $x^+ = (x_1^+, \dots, x_d^+)$  and  $x^- = (x_1^-, \dots, x_d^-)$ .
- (c) Give an expression for  $x$  in terms of  $x^+, x^-$ .
- (d) Give an expression for  $\|x\|_1$  without using any summations or absolute values.  
[Hint: Use  $x^+, x^-$  and the vector  $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{R}^d$ .]

2. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $S \subseteq \mathbb{R}$ . Consider the two optimization problems

$$\begin{array}{ll} \text{minimize}_{x \in \mathbb{R}} & |x| \\ \text{subject to} & f(x) \in S \end{array} \quad \text{and} \quad \begin{array}{ll} \text{minimize}_{a, b \in \mathbb{R}} & a + b \\ \text{subject to} & f(a - b) \in S \\ & a, b \geq 0. \end{array}$$

Solve the following questions.

- (a) If  $x$  in the first problem satisfies  $f(x) \in S$  show how to quickly compute  $(a, b)$  for the second problem with  $a + b = |x|$  and  $f(a - b) \in S$ .
  - (b) If  $a, b$  in the second problem satisfy  $f(a - b) \in S$ , show how to quickly compute an  $x$  for the first problem with  $|x| \leq a + b$  and  $f(x) \in S$ .
  - (c) Assume  $x$  is a minimizer for the first problem with minimum value  $p_1^*$  and  $(a, b)$  is a minimizer for the second problem with minimum  $p_2^*$ . Using the previous two parts, conclude that  $p_1^* = p_2^*$ .
3. Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $S \subseteq \mathbb{R}$  and consider the following optimization problem:

$$\begin{array}{ll} \text{minimize}_{x \in \mathbb{R}^d} & \|x\|_1 \\ \text{subject to} & f(x) \in S, \end{array}$$

where  $\|x\|_1 = \sum_{i=1}^d |x_i|$ . Give a new optimization problem with a linear objective function and the same minimum value. Show how to convert a solution to your new problem into a solution to the given problem. [Hint: Use the previous two problems.]