Machine Learning – Brett Bernstein

Week 2 Pre-Lecture: Concept Check Exercises

Optimization Prerequisites for Lasso

1. Given $a \in \mathbb{R}$ we define a^+, a^- as follows:

$$a^+ = \begin{cases} a & \text{if } a \geq 0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad a^- = \begin{cases} -a & \text{if } a < 0, \\ 0 & \text{otherwise.} \end{cases}$$

We call a^+ the *positive part* of a and a^- the *negative part* of a. Note that $a^+, a^- \geq 0$.

(a) Give an expression for a in terms of a^+, a^-.

(b) Give an expression for $|a|$ in terms of a^+, a^-.

For $x \in \mathbb{R}^d$ define $x^+ = (x_1^+, \ldots, x_d^+)$ and $x^- = (x_1^-, \ldots, x_d^-)$.

(c) Give an expression for x in terms of x^+, x^-.

(d) Give an expression for $\|x\|_1$ without using any summations or absolute values. [Hint: Use x^+, x^- and the vector $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^d$.]

Solution.

(a) $a = a^+ - a^-$

(b) $|a| = a^+ + a^-$

(c) $x = x^+ - x^-$

(d) $\|x\|_1 = \mathbf{1}^T(x^+ + x^-)$

2. Let $f : \mathbb{R} \to \mathbb{R}$ and $S \subseteq \mathbb{R}$. Consider the two optimization problems

$$\begin{array}{ll} \text{minimize}_{x \in \mathbb{R}} & |x| \\ \text{subject to} & f(x) \in S \end{array} \quad \text{and} \quad \begin{array}{ll} \text{minimize}_{a,b \in \mathbb{R}} & a + b \\ \text{subject to} & f(a-b) \in S \\ & a, b \geq 0. \end{array}$$

Solve the following questions.

(a) If x in the first problem satistfies $f(x) \in S$ show how to quickly compute (a, b) for the second problem with $a + b = |x|$ and $f(a - b) \in S$.

(b) If a, b in the second problem satisfy $f(a - b) \in S$, show how to quickly compute an x for the first problem with $|x| \leq a + b$ and $f(x) \in S$.

(c) Assume x is a minimizer for the first problem with minimum value p_1^* and (a, b) is a minimizer for the second problem with minimum p_2^*. Using the previous two parts, conclude that $p_1^* = p_2^*$.

Solution.

(a) Let $a = x^+$ and $b = x^-$. Then $a + b = |x|$ and $a - b = x$.

(b) Let $x = a - b$ and note that $|x| = |a - b| \leq |a| + |b| = a + b$.

(c) Part a) shows $p_2^* \leq p_1^*$ by letting $\hat{a} = x^+$ and $\hat{b} = x^-$. Part b) shows $p_1^* \leq p_2^*$ by letting $\hat{x} = a - b$.

3. Let $f : \mathbb{R}^d \to \mathbb{R}$, $S \subseteq \mathbb{R}$ and consider the following optimization problem:

$$\begin{array}{ll} \text{minimize}_{x \in \mathbb{R}^d} & \|x\|_1 \\ \text{subject to} & f(x) \in S, \end{array}$$

where $\|x\|_1 = \sum_{i=1}^d |x_i|$. Give a new optimization problem with a linear objective function and the same minimum value. Show how to convert a solution to your new problem into a solution to the given problem. [Hint: Use the previous two problems.]

Solution. Consider the minimization problem

$$\begin{array}{ll} \text{minimize}_{a,b \in \mathbb{R}^d} & \mathbf{1}^T(a + b) \\ \text{subject to} & f(a - b) \in S, \\ & a_i, b_i \geq 0 \quad \text{for } i = 1, \ldots, d. \end{array}$$

Let p_1^* be the minimum for the original problem, and p_2^* the minimum for our new problem. We first show $p_1^* = p_2^*$. Suppose x is a minimizer for the original problem and let $a = x^+$ and $b = x^-$. Then by the first question $\mathbf{1}^T(a+b) = \|x\|_1$ and $a - b = x$. This shows $p_2^* \leq p_1^*$. Next suppose (a, b) is a minimizer for our new problem, and let $x = a - b$. Then

$$\|x\|_1 = \|a - b\|_1 = \sum_{i=1}^{d} |a_i - b_i| \leq \sum_{i=1}^{d} |a_i| + |b_i| = \sum_{i=1}^{d} a_i + b_i = \mathbf{1}^T(a + b).$$

This proves $p_1^* \leq p_2^*$.

Finally, given a minimizer (a, b) for the new problem we recover a minimizer x for the original problem by letting $x = a - b$.