

Week 4 Lecture: Concept Check Exercises

Convexity

1. If $A, B \subseteq \mathbb{R}^n$ are convex, then $A \cap B$ is convex.
2. Let $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex. Show that $af + bg$ is convex if $a, b \geq 0$.
3. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex and differentiable. Prove that if $\nabla f(x) = 0$ then x is a global minimizer.
4. Prove that if $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is strictly convex and x is a global minimizer, then it is the unique global minimizer.
5. Prove that any affine function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is both convex and concave.
6. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex and let $g : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be affine. Then $f \circ g$ is convex.
7. (★★)
 - (a) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be convex. Show that f has one-sided left and right derivatives at every point.
 - (b) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex. Show that f has one-sided directional derivatives at every point.
 - (c) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex. Show that if x is not a minimizer of f then f has a descent direction at x (i.e., a direction whose corresponding one-sided directional derivative is negative).

Convex Optimization Problems

1. Suppose there are mn people forming m rows with n columns. Let a denote the height of the tallest person taken from the shortest people in each column. Let b denote the height of the shortest person taken from the tallest people in each row. What is the relationship between a and b ?
2. Let $x_1, \dots, x_n \in \mathbb{R}^d$ be given data. You want to find the center and radius of the smallest sphere that encloses all of the points. Express this problem as a convex optimization problem.
3. Suppose $x_1, \dots, x_n \in \mathbb{R}^d$ and $y_1, \dots, y_n \in \{-1, 1\}$. Here we look at y_i as the label of x_i . We say the data points are linearly separable if there is a vector $v \in \mathbb{R}^d$ and $a \in \mathbb{R}$ such that $v^T x_i > a$ when $y_i = 1$ and $v^T x_i < a$ for $y_i = -1$. Give a method for determining if the given data points are linearly separable.

4. Consider the Ivanov form of ridge regression:

$$\begin{aligned} & \text{minimize} && \|Ax - y\|_2^2 \\ & \text{subject to} && \|x\|_2^2 \leq r^2, \end{aligned}$$

where $r > 0$, $y \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$ are fixed.

- (a) What is the Lagrangian?
- (b) What do you get when you take the supremum of the Lagrangian over the feasible values for the dual variables?