Proportionality for Probability Distributions

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Abstract

Expressions involving the proportionality symbol $\propto$ are very common in mathematics, especially those involving probability models. It's an easy idea, and it can greatly simplify mathematical expressions. On the other hand, there is some ambiguity in the notation, that takes a bit of experience to get used to.

1 Basics

Let's start with an unambiguous example, with a single variable on each side:

$$f(x) \propto g(x)$$

means there exists a constant k such that

$$f(x) = kg(x)\ \forall x.$$

We can also have multiple variables, such as

$$f(x, y) \propto g(x, y),$$

which means $\exists k$ such that

$$f(x, y) = kg(x, y)\ \forall x, y.$$

2 Some variables fixed

Sometimes we want the proportionality constant to depend on one or more of the variables in our expression. For example, we can write the gamma density function as

$$p(x; \alpha, \beta) \propto x^{\alpha-1}e^{-\beta x}1(x > 0), \tag{2.1}$$

where $1(x > 0)$ is an **indicator function**[1] that takes the value 1 when $x > 0$ and is 0 otherwise. When we write proportionality in 2.1, we are thinking of each side as a function of x alone, with α and β held fixed. The proportionality constant will depend on α and β. Thus (2.1) means that there exists a function $k(\alpha, \beta)$ such that

$$p(x; \alpha, \beta) = k(\alpha, \beta) x^{\alpha-1} e^{-\beta x} 1(x > 0), \ \forall x, \alpha, \beta.$$

3 Recovering the proportionality constant by integration

Without additional information, we have no way to recover the proportionality constant in an expression like $f(x) \propto g(x)$. However, if we know something additional about $f(x)$, such as its integral or its value for a particular x, then we can determine k.

Let's consider the gamma density in (2.1). Since $p(x; \alpha, \beta)$ is a density in x, its integral over x must be 1:

$$\int_0^\infty k(\alpha, \beta) x^{\alpha-1} e^{-\beta x} \, dx = 1$$

Since the proportionality constant is, by definition, independent of x, it comes out of the integral and we can solve for it:

$$k(\alpha, \beta) = \left[\int_0^\infty x^{\alpha-1} e^{-\beta x} \, dx \right]^{-1} \tag{3.1}$$

If we can work out this integral, we are done. In this context, $k(\alpha, \beta)$ is called the **normalizing constant**, since multiplying the function by this factor makes the integral 1.

4 Recovering the proportionality constant by comparison

Suppose the integral in 3.1 is a bit too much for us. If we didn't know that the density was a gamma density, we could look through a table of known

[1] This indicator function designates the **support** of the density, which is the region where the density is nonzero.

probability densities and do some pattern matching. We would need to find a density proportional to $x^{\alpha-1}e^{-\beta x}1(x > 0)$. We would find that the gamma density is

$$p(x;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} 1(x > 0).$$

Thus $k(\alpha, \beta) = \beta^\alpha/\Gamma(\alpha)$.

Remark. It's **very important** to make sure that the supports of the densities are the same. With different supports, the normalizing constants would be different, and we cannot use this approach. Although here we use the indicator function to give the support directly in the expression for the density, sometimes the support for the distribution is given separately. For example, in Wikipedia the density for the Gamma distribution is given simply as

$$p(x;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x},$$

with the support indicated separately as $x \in (0, \infty)$.

Exercise: Justify this approach by showing that if we have two probability densities, $p(x)$ and $q(x)$, both of which have the same support and are proportional to the same function $f(x)$, then $p(x) = q(x)$.