

# Support Vector Machines

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# The SVM as a Quadratic Program

# The Margin

## Definition

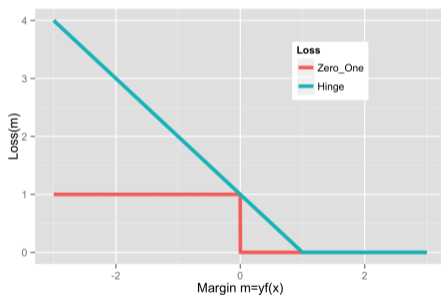
The **margin** (or **functional margin**) for predicted score  $\hat{y}$  and true class  $y \in \{-1, 1\}$  is  $y\hat{y}$ .

- The margin often looks like  $yf(x)$ , where  $f(x)$  is our score function.
- The margin is a measure of how **correct** we are.
- We want to **maximize the margin**.
- Most classification losses depend only on the margin.

(This is distinct from but related to **geometric margin** from lab.)

# Hinge Loss

- SVM/Hinge loss:  $\ell_{\text{Hinge}} = \max\{1 - m, 0\} = (1 - m)_+$
- Margin  $m = yf(x)$ ; “Positive part”  $(x)_+ = x1(x \geq 0)$ .



Hinge is a **convex, upper bound** on 0–1 loss. Not differentiable at  $m = 1$ . We have a “margin error” when  $m < 1$ .

# Support Vector Machine

- Hypothesis space  $\mathcal{F} = \{f(x) = w^T x + b \mid w \in \mathbf{R}^d, b \in \mathbf{R}\}$ .
- $\ell_2$  regularization (Tikhonov style)
- Loss  $\ell(m) = \max\{1 - m, 0\} = (1 - m)_+$
- The SVM prediction function is the solution to

$$\min_{w \in \mathbf{R}^d, b \in \mathbf{R}} \frac{1}{2} \|w\|^2 + \frac{c}{n} \sum_{i=1}^n \max(0, 1 - y_i [w^T x_i + b]).$$

# SVM Optimization Problem (Tikhonov Version)

The SVM prediction function is the solution to

$$\min_{w \in \mathbf{R}^d, b \in \mathbf{R}} \frac{1}{2} \|w\|^2 + \frac{c}{n} \sum_{i=1}^n \max(0, 1 - y_i [w^T x_i + b]).$$

- unconstrained optimization
- not differentiable because of the max (right at the border of a margin error)
- Can we reformulate into a differentiable problem?

# SVM Optimization Problem

- The SVM optimization problem is equivalent to

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2} \|w\|^2 + \frac{c}{n} \sum_{i=1}^n \xi_i \\ \text{subject to} \quad & \xi_i \geq \max(0, 1 - y_i [w^T x_i + b]). \end{aligned}$$

- Which is equivalent to

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2} \|w\|^2 + \frac{c}{n} \sum_{i=1}^n \xi_i \\ \text{subject to} \quad & \xi_i \geq (1 - y_i [w^T x_i + b]) \text{ for } i = 1, \dots, n \\ & \xi_i \geq 0 \text{ for } i = 1, \dots, n \end{aligned}$$

# SVM as a Quadratic Program

- The SVM optimization problem is equivalent to

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2} \|w\|^2 + \frac{c}{n} \sum_{i=1}^n \xi_i \\ \text{subject to} \quad & -\xi_i \leq 0 \text{ for } i = 1, \dots, n \\ & (1 - y_i [w^T x_i + b]) - \xi_i \leq 0 \text{ for } i = 1, \dots, n \end{aligned}$$

- Differentiable objective function
- $n + d + 1$  unknowns and  $2n$  affine constraints.
- A quadratic program that can be solved by any off-the-shelf QP solver.
- Let's learn more by examining the dual.



# The SVM Dual Problem

## SVM Lagrange Multipliers

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2} \|w\|^2 + \frac{c}{n} \sum_{i=1}^n \xi_i \\ \text{subject to} \quad & -\xi_i \leq 0 \text{ for } i = 1, \dots, n \\ & (1 - y_i [w^T x_i + b]) - \xi_i \leq 0 \text{ for } i = 1, \dots, n \end{aligned}$$

Lagrange Multiplier	Constraint
$\lambda_i$	$-\xi_i \leq 0$
$\alpha_i$	$(1 - y_i [w^T x_i + b]) - \xi_i \leq 0$

$$L(w, b, \xi, \alpha, \lambda) = \frac{1}{2} \|w\|^2 + \frac{c}{n} \sum_{i=1}^n \xi_i + \sum_{i=1}^n \alpha_i (1 - y_i [w^T x_i + b] - \xi_i) + \sum_{i=1}^n \lambda_i (-\xi_i)$$

# SVM Lagrangian

- The Lagrangian for this formulation is

$$\begin{aligned}
 & L(w, b, \xi, \alpha, \lambda) \\
 = & \frac{1}{2} \|w\|^2 + \frac{c}{n} \sum_{i=1}^n \xi_i + \sum_{i=1}^n \alpha_i (1 - y_i [w^T x_i + b] - \xi_i) - \sum_i \lambda_i \xi_i \\
 = & \frac{1}{2} w^T w + \sum_{i=1}^n \xi_i \left( \frac{c}{n} - \alpha_i - \lambda_i \right) + \sum_{i=1}^n \alpha_i (1 - y_i [w^T x_i + b]).
 \end{aligned}$$

- Primal and dual:

$$\begin{aligned}
 p^* &= \inf_{w, \xi, b} \sup_{\alpha, \lambda \geq 0} L(w, b, \xi, \alpha, \lambda) \\
 &\geq \sup_{\alpha, \lambda \geq 0} \inf_{w, b, \xi} L(w, b, \xi, \alpha, \lambda) = d^*
 \end{aligned}$$

- Do we have  $p^* = d^*$ ?

## Strong Duality by Slater's constraint qualification

- The SVM optimization problem:

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2} \|w\|^2 + \frac{c}{n} \sum_{i=1}^n \xi_i \\ \text{subject to} \quad & -\xi_i \leq 0 \text{ for } i = 1, \dots, n \\ & (1 - y_i [w^T x_i + b]) - \xi_i \leq 0 \text{ for } i = 1, \dots, n \end{aligned}$$

- Convex problem + affine constraints  $\implies$  strong duality iff problem is feasible
- Constraints are satisfied by  $w = b = 0$  and  $\xi_i = 1$  for  $i = 1, \dots, n$ ,
  - so **we have strong duality**  $\implies$

$$\begin{aligned} p^* &= \inf_{w, \xi, b} \sup_{\alpha, \lambda \geq 0} L(w, b, \xi, \alpha, \lambda) \\ &= \sup_{\alpha, \lambda \geq 0} \inf_{w, b, \xi} L(w, b, \xi, \alpha, \lambda) = d^* \end{aligned}$$

# SVM Dual Function

- Lagrange dual is the inf over primal variables of the Lagrangian:

$$\begin{aligned}
 g(\alpha, \lambda) &= \inf_{w, b, \xi} L(w, b, \xi, \alpha, \lambda) \\
 &= \inf_{w, b, \xi} \left[ \frac{1}{2} w^T w + \sum_{i=1}^n \xi_i \left( \frac{c}{n} - \alpha_i - \lambda_i \right) + \sum_{i=1}^n \alpha_i (1 - y_i [w^T x_i + b]) \right]
 \end{aligned}$$

- Taking inf of convex and differentiable function of  $w, b, \xi$ .
  - Quadratic in  $w$  and linear in  $\xi$  and  $b$ .
- Thus optimal point iff  $\partial_w L = 0 \partial_b L = 0 \partial_\xi L = 0$
- Note:  $g(\alpha, \lambda) = -\infty$  when  $\frac{c}{n} - \alpha_i - \lambda_i \neq 0$ . (send  $\xi_i \rightarrow \pm\infty$ ). This inf is NOT an optimum because it is never attained.

## SVM Dual Function: First Order Conditions

Lagrange dual function is the inf over primal variables of  $L$ :

$$g(\alpha, \lambda) = \inf_{w, b, \xi} L(w, b, \xi, \alpha, \lambda)$$

$$= \inf_{w, b, \xi} \left[ \frac{1}{2} w^T w + \sum_{i=1}^n \xi_i \left( \frac{c}{n} - \alpha_i - \lambda_i \right) + \sum_{i=1}^n \alpha_i (1 - y_i [w^T x_i + b]) \right]$$

$$\partial_w L = 0 \iff w - \sum_{i=1}^n \alpha_i y_i x_i = 0 \iff w = \sum_{i=1}^n \alpha_i y_i x_i$$

$$\partial_b L = 0 \iff - \sum_{i=1}^n \alpha_i y_i = 0 \iff \sum_{i=1}^n \alpha_i y_i = 0$$

$$\partial_{\xi_i} L = 0 \iff \frac{c}{n} - \alpha_i - \lambda_i = 0 \iff \alpha_i + \lambda_i = \frac{c}{n}$$

# SVM Dual Function

- Substituting these conditions back into  $L$ , the second term disappears.
- First and third terms become

$$\frac{1}{2}w^T w = \frac{1}{2} \sum_{i,j=1}^n \alpha_i \alpha_j y_i y_j x_i^T x_j$$

$$\sum_{i=1}^n \alpha_i (1 - y_i [w^T x_i + b]) = \sum_{i=1}^n \alpha_i - \sum_{i,j=1}^n \alpha_i \alpha_j y_i y_j x_j^T x_i - b \underbrace{\sum_{i=1}^n \alpha_i y_i}_{=0}$$

- Putting it together, the dual function is

$$g(\alpha, \lambda) = \begin{cases} \sum_{i=1}^n \alpha_i - \frac{1}{2} \sum_{i,j=1}^n \alpha_i \alpha_j y_i y_j x_j^T x_i & \begin{array}{l} \sum_{i=1}^n \alpha_i y_i = 0 \\ \alpha_i + \lambda_i = \frac{c}{n}, \text{ all } i \end{array} \\ -\infty & \text{otherwise.} \end{cases}$$

## SVM Dual Problem

- The **dual function** is

$$g(\alpha, \lambda) = \begin{cases} \sum_{i=1}^n \alpha_i - \frac{1}{2} \sum_{i,j=1}^n \alpha_i \alpha_j y_i y_j x_j^T x_i & \sum_{i=1}^n \alpha_i y_i = 0 \\ -\infty & \alpha_i + \lambda_i = \frac{c}{n}, \text{ all } i \\ & \text{otherwise.} \end{cases}$$

- The **dual problem** is  $\sup_{\alpha, \lambda \geq 0} g(\alpha, \lambda)$ :

$$\begin{aligned} \sup_{\alpha, \lambda} \quad & \sum_{i=1}^n \alpha_i - \frac{1}{2} \sum_{i,j=1}^n \alpha_i \alpha_j y_i y_j x_j^T x_i \\ \text{s.t.} \quad & \sum_{i=1}^n \alpha_i y_i = 0 \\ & \alpha_i + \lambda_i = \frac{c}{n} \quad \alpha_i, \lambda_i \geq 0, \quad i = 1, \dots, n \end{aligned}$$