Bayesian Methods

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March 26, 2019

Contents

Classical Statistics

Parametric Family of Densities

- A **parametric family of densities** is a set

$$\{p(y \mid \theta) : \theta \in \Theta\},$$

  - where $p(y \mid \theta)$ is a density on a **sample space** $\mathcal{Y}$, and
  - θ is a **parameter** in a [finite dimensional] **parameter space** Θ.

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- This is the common starting point for a treatment of classical or Bayesian statistics.

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- (In more advanced, measure-theoretic treatments, they are each considered densities w.r.t. different base measures.)

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- Instead of θ, we have data $\mathcal{D}$: $y_1, \ldots, y_n$ sampled i.i.d. $p(y \mid \theta)$.
- Statistics is about how to get by with $\mathcal{D}$ in place of θ.

Point Estimation

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- A good point estimator will have $\hat{\theta} \approx \theta$.

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  - **Efficiency:** (Roughly speaking) $\hat{\theta}_n$ is as accurate as we can get from a sample of size n.
- **Maximum likelihood estimators** are consistent and efficient under reasonable conditions.

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- For fixed θ, $p(\mathcal{D} \mid \theta)$ is a density function on $\mathcal{Y}^n$.
- For fixed $\mathcal{D}$, the function $\theta \mapsto p(\mathcal{D} \mid \theta)$ is called the **likelihood function:**

$$L_{\mathcal{D}}(\theta) := p(\mathcal{D} \mid \theta).$$

Maximum Likelihood Estimation

Definition

The **maximum likelihood estimator (MLE)** for θ in the model $\{p(y \mid \theta) : \theta \in \Theta\}$ is

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- Maximum likelihood is just one approach to getting a point estimator for θ.
- **Method of moments** is another general approach one learns about in statistics.
- Later we'll talk about **MAP** and **posterior mean** as approaches to point estimation.
  - These arise naturally in Bayesian settings.

Coin Flipping: Setup

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- Note that every $\theta \in \Theta$ gives us a different probability model for a coin.

Coin Flipping: Likelihood function

- Data $\mathcal{D} = (H, H, T, T, T, T, T, H, \ldots, T)$
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- This is the probability of getting the flips in the order they were received.

Coin Flipping: MLE

- As usual, easier to maximize the log-likelihood function:

$$\begin{aligned}\hat{\theta}_{\text{MLE}} &= \underset{\theta \in \Theta}{\arg\max} \log L_{\mathcal{D}}(\theta) \\ &= \underset{\theta \in \Theta}{\arg\max} \left[n_h \log \theta + n_t \log(1-\theta) \right]\end{aligned}$$

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- First order condition:

$$\begin{aligned}\frac{n_h}{\theta} - \frac{n_t}{1-\theta} &= 0\\ \iff \theta &= \frac{n_h}{n_h+n_t}.\end{aligned}$$

- So $\hat{\theta}_{\text{MLE}}$ is the empirical fraction of heads.

Bayesian Statistics: Introduction

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- A prior reflects our belief about θ, **before seeing any data**..

A Bayesian Model

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  1. A parametric family of densities
  $$\{p(\mathcal{D} \mid \theta) \mid \theta \in \Theta\}.$$
  2. A **prior distribution** $p(\theta)$ on parameter space Θ.
- Putting pieces together, we get a joint density on θ and $\mathcal{D}$:

$$p(\mathcal{D}, \theta) = p(\mathcal{D} \mid \theta)p(\theta).$$

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- Posterior represents the **rationally "updated" belief** about θ, after seeing $\mathcal{D}$.

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- Then both sides are densities on Θ and we can write

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- Where $\propto$ means we've dropped factors independent of θ.

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- Need a prior distribution $p(\theta)$ on $\Theta = (0,1)$.
- A distribution from the Beta family will do the trick...

Coin Flipping: Beta Prior

- **Prior:**

$$\begin{aligned} \theta &\sim \text{Beta}(\alpha, \beta) \\ p(\theta) &\propto \theta^{\alpha-1}(1-\theta)^{\beta-1} \end{aligned}$$

Figure by Horas based on the work of Krishnavedala (Own work) [Public domain], via Wikimedia Commons http://commons.wikimedia.org/wiki/File:Beta_distribution_pdf.svg.

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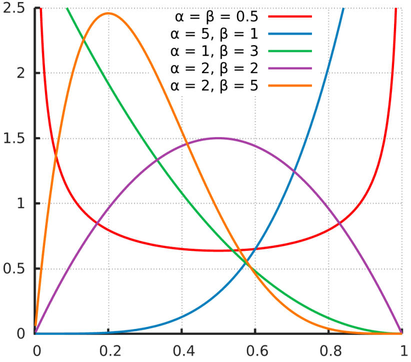

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- **Mode of Beta distribution:**

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\arg\max_{\theta} p(\theta) = \frac{h-1}{h+t-2}
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for $h,t>1$.

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- **Interpretation**:
  - Prior initializes our counts with h heads and t tails.
  - Posterior increments counts by observed n_h and n_t.

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- The family of all probability distributions is conjugate to any parametric model. [Trvially]

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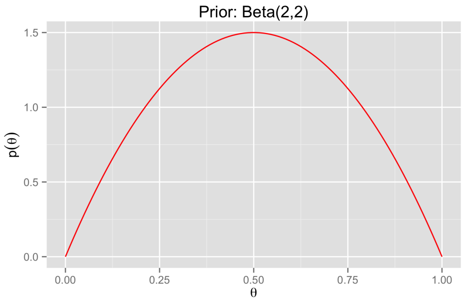

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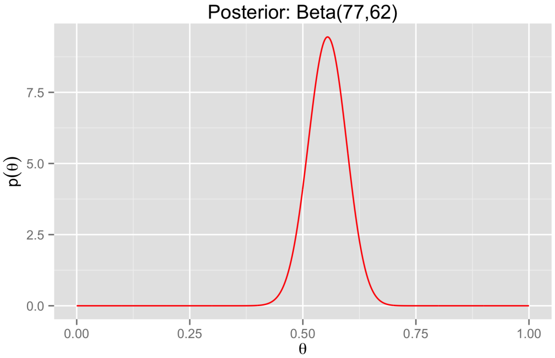
  - $\hat{\theta}_{\text{MLE}} = \frac{75}{75+60} \approx 0.556$
- **Posterior distribution:** $\theta \mid \mathcal{D} \sim \textbf{Beta}(77, 62)$**:**



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- Common options:
  - **posterior mean** $\hat{\theta} = \mathbb{E}[\theta \mid \mathcal{D}]$
  - **maximum a posteriori (MAP) estimate** $\hat{\theta} = \arg\max_\theta p(\theta \mid \mathcal{D})$
    - Note: this is the **mode** of the posterior distribution

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- Extract "**credible set**" for θ (a Bayesian confidence interval).
    - e.g. Interval $[a, b]$ is a 95% **credible set** if

$$\mathbb{P}\left(\theta \in [a, b] \mid \mathcal{D}\right) \geqslant 0.95$$

What else can we do with a posterior?

- Look at it.
- Extract "**credible set**" for θ (a Bayesian confidence interval).
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- The most "Bayesian" approach is **Bayesian decision theory**:
  - Choose a loss function.
  - Find action **minimizing expected risk w.r.t. posterior**

Bayesian Decision Theory

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- Ingredients:
  - **Parameter space** Θ.
  - **Prior**: Distribution $p(\theta)$ on Θ.
  - **Action space** $\mathcal{A}$.
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  - It's the **expected loss under the posterior.**
- A **Bayes action** a^* is an action that minimizes posterior risk:

$$
r(a^*) = \min_{a \in \mathcal{A}} r(a)
$$

Bayesian Point Estimation

- General Setup:
  - Data $\mathcal{D}$ generated by $p(y \mid \theta)$, for unknown $\theta \in \Theta$.

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Bayesian Point Estimation: Square Loss

- Find **action** $\hat{\theta} \in \Theta$ that minimizes **posterior risk**

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- **Bayes action** for **square loss** is the posterior mean.

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  - That is, the median of the distribution $p(\theta \mid \mathcal{D})$.
  - Show with approach similar to what was used in Homework #1.

Bayesian Point Estimation: Zero-One Loss

- Suppose Θ is discrete (e.g. $\Theta = \{\text{english}, \text{french}\}$)
- **Zero-one loss:** $\ell(\theta, \hat{\theta}) = 1(\theta \neq \hat{\theta})$

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- This $\hat{\theta}$ is called the **maximum a posteriori (MAP)** estimate.
- The MAP estimate is the **mode** of the posterior distribution.

Summary

Recap and Interpretation

- Prior represents belief about θ before observing data $\mathcal{D}$.
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    - **family of distributions**, indexed by Θ, and the
    - **prior distribution** on Θ
  - For decision making, need a **loss function**.
  - Everything after that is **computation**.

The Bayesian Method

1. **Define the model**:
   - Choose a parametric family of densities:

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