Week 4 Lecture: Concept Check Exercises

Convexity

1. If $A, B \subseteq \mathbb{R}^n$ are convex, then $A \cap B$ is convex.

2. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be convex. Show that $af + bg$ is convex if $a, b \geq 0$.

3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and differentiable. Prove that if $\nabla f(x) = 0$ then $x$ is a global minimizer.

4. Prove that if $f : \mathbb{R}^n \to \mathbb{R}$ is strictly convex and $x$ is a global minimizer, then it is the unique global minimizer.

5. Prove that any affine function $f : \mathbb{R}^n \to \mathbb{R}$ is both convex and concave.

6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and let $g : \mathbb{R}^m \to \mathbb{R}^n$ be affine. Then $f \circ g$ is convex.

7. (**)
   
   (a) Let $f : \mathbb{R} \to \mathbb{R}$ be convex. Show that $f$ has one-sided left and right derivatives at every point.

   (b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Show that $f$ has one-sided directional derivatives at every point.

   (c) Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Show that if $x$ is not a minimizer of $f$ then $f$ has a descent direction at $x$ (i.e., a direction whose corresponding one-sided directional derivative is negative).

Convex Optimization Problems

1. Suppose there are $mn$ people forming $m$ rows with $n$ columns. Let $a$ denote the height of the tallest person taken from the shortest people in each column. Let $b$ denote the height of the shortest person taken from the tallest people in each row. What is the relationship between $a$ and $b$?

2. Let $x_1, \ldots, x_n \in \mathbb{R}^d$ be given data. You want to find the center and radius of the smallest sphere that encloses all of the points. Express this problem as a convex optimization problem.

3. Suppose $x_1, \ldots, x_n \in \mathbb{R}^d$ and $y_1, \ldots, y_n \in \{-1, 1\}$. Here we look at $y_i$ as the label of $x_i$. We say the data points are linearly separable if there is a vector $v \in \mathbb{R}^d$ and $a \in \mathbb{R}$ such that $v^T x_i > a$ when $y_i = 1$ and $v^T x_i < a$ for $y_i = -1$. Give a method for determining if the given data points are linearly separable.
4. Consider the Ivanov form of ridge regression:

\[
\begin{align*}
\text{minimize} \quad & \|Ax - y\|_2^2 \\
\text{subject to} \quad & \|x\|_2^2 \leq r^2,
\end{align*}
\]

where \(r > 0\), \(y \in \mathbb{R}^m\) and \(A \in \mathbb{R}^{m \times n}\) are fixed.

(a) What is the Lagrangian?

(b) What do you get when you take the supremum of the Lagrangian over the feasible values for the dual variables?