Completing the Square

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1 Completing the Square (Univariate)

You may remember from elementary algebra the notion of "completing the square." Given a variable $x \in \mathbf{R}$, consider the second order polynomial in x:

$$x^2 + bx + c. \tag{1.1}$$

We would like to rewrite this expression in the form

$$(x+d)^2 + k, \tag{1.2}$$

since this form is often easier to work with. Note that (1.2) has only a squared term and a constant term, but no linear term. Expanding (1.2) we get

$$(x+d)^2 + k = x^2 + 2dx + d^2 + k.$$

Equating corresponding terms between (1.1) and (1.2), we can simply take $d = b/2$ and $k = c - d^2 = c - b^2/4$. Putting it together, we get

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right). \tag{1.3}$$

Thus any expression of the form (1.1) can also be written in the form of (1.2), and the recipe is given in (1.3).

It turns out that we can generalize this process to quadratic expressions involving matrices and vectors.

2 Completing the Square (Multivariate)

Proposition 1. *For any vectors $x, b \in \mathbf{R}^d$ and symmetric invertible matrix $M \in \mathbf{R}^{d \times d}$, we have*

$$x^T M x - 2b^T x = \left(x - M^{-1}b\right)^T M (x - M^{-1}b) - b^T M^{-1} b \tag{2.1}$$

Proof. This is immediate just by multiplying out the quadratic form:

$$\begin{aligned}\left(x - M^{-1}b\right)^T M(x - M^{-1}b) &= x^T Mx - 2b^T M^{-1} Mx + b^T M^{-1} M M^{-1} b \\ &= x^T Mx - 2b^T x + b^T M^{-1} b\end{aligned}$$

Rearranging gives (2.1). □

3 Examples

Proposition 2. *(Sum of two quadratic forms in x) Suppose $f(x)$ is the sum of two quadratic forms in x:*

$$f(x) = (x-\mu)^T \Sigma^{-1} (x-\mu) + (x-\theta)^T V^{-1} (x-\theta)$$

Then we can write f as a single quadratic form plus a constant term, independent of x:

$$f(x) = \left(x - M^{-1}b\right)^T M(x - M^{-1}b) - b^T M^{-1} b + R,$$

where $M = \Sigma^{-1} + V^{-1}$, $b = \Sigma^{-1}\mu + V^{-1}\theta$, and $R = \theta' V^{-1}\theta + \mu' \Sigma^{-1} \mu$.

Proof. Expanding out the two quadratic forms in the expression for $f(x)$, we have

$$\begin{aligned}(x-\mu)^T \Sigma^{-1} (x-\mu) &= x^T \Sigma^{-1} x - 2\mu^T \Sigma^{-1} x + \mu^T \Sigma^{-1} \mu \\ (x-\theta)^T V^{-1} (x-\theta) &= x^T V^{-1} x - 2\theta^T V^{-1} x + \theta^T V^{-1} \theta.\end{aligned}$$

Adding these expressions together and combining terms, we get

$$\begin{aligned}f(x) &= x^T \left(\Sigma^{-1} + V^{-1}\right) x - 2\left(\mu^T \Sigma^{-1} + \theta^T V^{-1}\right) x + R \\ &= x^T Mx - 2b^T x + R \\ &= \left(x - M^{-1}b\right)^T M(x - M^{-1}b) - b^T M^{-1} b + R,\end{aligned}$$

where the last equation follows from Proposition 1. □